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p -good groups and P -good modules (Cohomology theory of finite groups)

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p -good groups and P -good modules

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This is a preliminary report of my joint work with Professors T.Wada, M.Murai and A.Hanaki. Let us fix our notation.

G = a finite group

k = an algebraically closed field of characteristic $p > 0$

$P_G(S)$ = the projective cover of a simple kG -module S

1_G = the trivial kG -module

$P(G) = P_G(1_G)$, the projective cover of 1_G

Definition 1. A finite group G is called p -good if for every simple kG -module S the following condition holds;

$$P(G) \otimes S \cong m_S P_G(S)$$

with some positive integer m_S .

We are interested in group structures of p -good groups. If G is a p -group or a p' -group, then G is clearly p -good. G is p -good if and only if $G/O_p(G)$ is p -good. S_4 , symmetric group of degree 4, is 2-good but not 3-good. We have two fundamental questions on group structures of p -good groups.

Question 1. If G is p -good, then is it true that G is p -solvable?

Question 2. Let G be p -solvable and p -good. Is it true that the p -length of G is bounded?

Now we will define good kG -modules.

Definition 2. Let M be a kG -module and P a projective kG -module. We call M P -good if a simple kG -module S is a composition factor of M whenever $P_G(S)$ is a direct factor of $P \otimes M$. M is called good if M is $P(G)$ -good.

By definition, M is $P_1 \oplus P_2$ -good if and only if M is P_i -good ($i = 1, 2$). Note that the following statements are equivalent to each other;

- (1) G is p -good,
- (2) M is good for every (finitely generated) kG -module M ,
- (3) S is good for every simple kG -module S .

The following lemma is well-known.

Lemma 1. *Let $H \triangleleft G$, S be a simple kG -module and X a simple kH -module. Then X is a composition factor of S_H if and only if S is a composition factor of X^G . Here S_H is the restriction of S to H , and X^G is the induction of X to G .*

Using Lemma 1 we can prove the following propositions.

Proposition 1. *Let $H \triangleleft G$, M be a kG -module and P a projective kG -module. If M is P -good, then M_H is P_H -good.*

Proposition 2. *Let $H \triangleleft G$, N be a kH -module and Q a projective kH -module. If N is Q^x -good for all $x \in G$, then N^G is Q^G -good.*

Since $P(H)$ is a direct summand of $P(G)_H$ and $P(G)$ is a direct summand of $P(H)^G$, we have the following

Corollary 3. *Let $H \triangleleft G$, M be a kG -module and N a kH -module.*

- (1) *If M is good, then M_H is good.*
- (2) *If N is good, then N^G is good.*

When the index $|G : H|$ is a power of p , the converse of Proposition 1 also holds.

Proposition 4. *Let H be a normal subgroup of G with p -power index. Let M be a kG -module and P a projective kG -module. Then M is P -good, if and only if M_H is P_H -good.*

Corollary 5. *Let H be a normal subgroup of G with p -power index. Let M be a kG -module. Then M is good, if and only if M_H is good.*

Corollary 5 yields the following two results on p -good groups.

Corollary 6. *Let H be a normal subgroup of G with p -power index. If H is p -good then G is p -good.*

Corollary 7. *If G is a p -solvable group with $G = O_{p,p',p}(G)$, then G is p -good.*

Question 3. Does the converse of Corollary 6 hold?

Question 4. When G is p -good and p -solvable with p -length 1, describe the group structure of G . Recall that S_4 is 3-solvable with 3-length 1, but not 3-good.

For p -solvable groups G , we have the following criterion for a simple kG -module S to be good.

Proposition 8. *Let G be a p -solvable group with a p -complement L . Let S be a simple kG -module. Then the following inequality holds;*

$$\dim_k \text{End}_{kL}(S_L) \leq (\dim_k S)_p.$$

Moreover, the following statements are equivalent;

- (1) S is good,
- (2) $\dim_k \text{End}_{kL}(S_L) = (\dim_k S)_p$,
- (3) $\text{Hom}_{kL}(S_L, T_L) = 0$ for every simple kG -module T with $T \not\cong S$.

Corollary 9. *Let G be a p -solvable group. Then the following statements are equivalent;*

- (1) G is p -good,
- (2) $\dim_k \text{End}_{kL}(S_L) = (\dim_k S)_p$ for every simple kG -module S ,
- (3) $\text{Hom}_{kL}(S_L, T_L) = 0$ for all simple kG -modules S, T with $S \not\cong T$.